There are 8 questions with total 126 points in this exam.

1. Let $\{f_n\}$ be a sequence of functions with domain $D \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$.
   (a) (4 points) Define what it means to say that $\{f_n\}$ is **pointwise convergent to** f **on** D.
   (b) (4 points) Define what it means to say that $\{f_n\}$ is **uniformly convergent to** f **on** D.
   (c) (4 points) Define what it means to say that $\{f_n\}$ is **Not uniformly convergent to** f **on** D.

2. Let f be a function with domain $D \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$.
   (a) (4 points) Define what it means to say that f is a **continuous** function on D.
   (b) (4 points) Define what it means to say that f is a **uniformly continuous** function on D.
   (c) (4 points) Define what it means to say that f is a **Lipschitz** function on D..

3. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x^2 - y^2, 2xy)$. For each $(x, y) \neq (0, 0)$, show that there is an open neighborhood U of (x, y) such that f has a (local) C^1 inverse defined on $f(U)$.

Solution: Since f is smooth and $\det Df = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \neq 0$ for each $(x, y) \neq (0, 0)$, there is an open neighborhood U of (x, y) on which f has a (local) C^1 inverse defined on $f(U)$ by the Inverse Function Theorem.

4. (10 points) In the system

$$\begin{aligned} 3x + 2y + z^2 + u + v^2 &= 0 \\ 4x + 3y + z + u^2 + v + w + 2 &= 0 \\ x + z + w + u^2 + 2 &= 0, \end{aligned}$$

discuss the solvability for u, v, w in terms of x, y, z near the point $(x, y, z, u, v, w) = (0, 0, 0, 0, 0, -2)$.

Solution: Let $F(x, y, z, u, v, w) = (3x + 2y + z^2 + u + v^2\ ,\ 4x + 3y + z + u^2 + v + w + 2\ ,\ x + z + w + u^2 + 2)$.
Direct computation gives that $DF|_{(0,0,0,0,0,-2)} = \begin{bmatrix} 3 & 2 & 2z & 1 & 2v & 0 \\ 4 & 3 & 1 & 2u & 1 & 1 \\ 1 & 0 & 1 & 2u & 0 & 1 \end{bmatrix}_{(0,0,0,0,0,-2)} = \begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.
Since $\det DF|_{(u,v,w)=(0,0,-2)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$, one can solve for u, v, w in terms of x, y, z near the point $(x, y, z, u, v, w) = (0, 0, 0, 0, 0, -2)$ by the Implicit Function Theorem.

5. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) = (x + y^3, xy, y + y^2)$. Is the range of f a two-dimensional surface or a one-dimensional curve near $(0, 0)$?

Solution: Since$DF|_{(0,0)} = \begin{bmatrix} 1 & 3y^2 \\ y & x \\ 0 & 1 + 2y \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ has rank 2, the range a smooth surface near $(0, 0)$.

6. Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by $f(x_1, x_2, y_1, y_2, y_3) = (2e^{x_1} + x_2 y_1 - 4y_2 + 3\ ,\ x_2 \cos x_1 - 6x_1 + 2y_1 - y_3)$ so that $f(0, 1, 3, 2, 7) = (0, 0)$ and $Df(0, 1, 3, 2, 7) = \begin{pmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{pmatrix}$.

(a) (8 points) Show that we can solve for $(x_1, x_2) = g(y_1, y_2, y_3)$ i.e. solve for x_1, x_2 in terms of y_1, y_2, y_3, near $(y_1, y_2, y_3) = (3, 2, 7)$.

Solution: Since $\det D_x f|_{(x_1,x_2)=(0,1)} = \begin{vmatrix} 2 & 3 \\ -6 & 1 \end{vmatrix} = 20 \neq 0$, one can solve for x_1, x_2 in terms of y_1, y_2, y_3, i.e. there exists a C^1 function g such that $(x_1, x_2) = g(y) = g(y_1, y_2, y_3)$, for those $(x_1, x_2, y_1, y_2, y_3)$ satisfying that $f(x_1, x_2, y_1, y_2, y_3) = (0, 0)$ near $(y_1, y_2, y_3) = (3, 2, 7)$ by the Implicit Function Theorem.

(b) (10 points) Show that $Dg(3, 2, 7) = -\dfrac{1}{20}\begin{pmatrix} 1 & -3 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

Solution: Part (a) implies that $f(x_1, x_2, y_1, y_2, y_3) = f(g(y), y)$ near $(y_1, y_2, y_3) = (3, 2, 7)$. Using chain rule and differentiating $f(x, y) = f(g(y), y)$ with respect to y_i, for $i = 1, 2, 3$, we obtain that

$f_{i,x_1}\dfrac{\partial g_1}{\partial y_j} + f_{i,x_2}\dfrac{\partial g_2}{\partial y_j} + \dfrac{\partial f_i}{\partial y_j} = 0$ for $i = 1, 2$ and $j = 1, 2, 3$.

$$\Leftrightarrow \begin{bmatrix} f_{i,x_1} & f_{i,x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_i}{\partial y_1} & \frac{\partial f_i}{\partial y_2} & \frac{\partial f_i}{\partial y_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \text{ for } i = 1, 2.$$

$$\Leftrightarrow \begin{bmatrix} f_{1,x_1} & f_{1,x_2} \\ f_{2,x_1} & f_{2,x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \end{bmatrix} = 0$$

$$\Leftrightarrow Dg = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix} = -\begin{bmatrix} f_{1,x_1} & f_{1,x_2} \\ f_{2,x_1} & f_{2,x_2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \end{bmatrix}.$$

Evaluating at $(0, 1, 3, 2, 7)$, we obtain

$$Dg(3, 2, 7) = -\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = -\frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

7. (a) (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{1 + x^2}$. Prove that f is **uniformly continuous on** $\mathbb{R}$.

[**Hint:** You may use the Mean Value Theorem and the inequality $\dfrac{2ab}{a^2 + b^2} \leq 1$ when $a^2 + b^2 \neq 0$.]

Solution: For each $x, y \in \mathbb{R}$, by the Mean Value Theorem, $|f(x) - f(y)| = |f'(z)(x - y)| = |\dfrac{2z}{(1 + z^2)^2}||x - y|$ holds for some z lying between x and y. Using the inequality $\dfrac{2ab}{a^2 + b^2} \leq 1$ when $a^2 + b^2 \neq 0$, we have $|f(x) - f(y)| \leq \dfrac{1}{1 + z^2}|x - y| \leq |x - y|$ for each $x, y \in \mathbb{R}$. Hence, f is uniformly continuous on $\mathbb{R}$ since it is Lipschitz there.

(b) (8 points) Let $g(x) = \tan x$ for $x \in [0, \frac{\pi}{2})$. Prove that g is **Not Lipschitz on** $[0, \frac{\pi}{2})$.

Solution: For each $x, y \in [0, \frac{\pi}{2})$, by the Mean Value Theorem, $|\tan x - \tan y| = \sec^2 z|x - y|$ holds for some z lying between x and y. Since $\lim\limits_{x,y \to (\pi/2)^-} \sec^2 z = \lim\limits_{z \to (\pi/2)^-} \sec^2 z = \infty$, g is not Lipschitz on $[0, \frac{\pi}{2})$.

(c) (8 points) Let $f(x) = \dfrac{1}{2}\left(x + \dfrac{2}{x}\right)$ for $x \in S = [1, \infty)$. Prove that f is a contraction mapping of S, and find the fixed point of f.

Solution: The Mean Value Theorem implies that $|f(x) - f(y)| = |f'(z)(x-y)|$ holds for some z lying between $x, y \in [1, \infty)$. Since $|f'(z)| = |\frac{1}{2} - \frac{1}{x^2}| \leq \frac{1}{2} < 1$, we obtain that $|f(x) - f(y)| \leq \frac{1}{2}|x-y|$ holds for all $x, y \in [1, \infty)$ which implies that f is a contraction mapping of S.
A point $x \in S$ is a fixed point of f if $f(x) = x \Leftrightarrow x^2 = 2$, $x \in S \Leftrightarrow x = \sqrt{2}$.

8. Let $\{f_n\}$ be a sequence of functions defined by $f_n(x) = \dfrac{nx}{1+nx^2}$ for each $x \in [0,1]$.
   (a) (6 points) Find the limit $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$. [**Hint:** $x \in [0,1] = \{0\} \cup (0,1]$.]

   Solution: $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ \dfrac{1}{x} & \text{if } x \in (0,1]. \end{cases}$

   (b) (6 points) Show that the convergence in **Not** uniform on $[0,1]$.

   Solution: Since f is not continuous at $x = 0$ the convergence is not uniform on $[0,1]$.

9. Let $\{f_n\}$ be a sequence of functions defined by $f_n(x) = \sqrt{n}x^n(1-x)$ for each $x \in [0,1]$.
   (a) (6 points)Find $\max_{x\in[0,1]} f_n(x)$.

   Solution: Since $f_n'(x) = n\sqrt{n}\,x^{n-1}(1-x) - \sqrt{n}\,x^n = \sqrt{n}x^{n-1}\big[n - (n+1)x\big] = 0$ when $x = \dfrac{n}{n+1}$, we obtain that $\max_{x\in[0,1]} f_n(x) = f_n(\frac{n}{n+1}) = \sqrt{n}\,(\frac{n}{n+1})^n(1 - \frac{n}{n+1}) = \dfrac{\sqrt{n}}{n+1}(1 - \frac{1}{n+1})^n$.

   (b) (6 points) Find the limit $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$. [**Hint:** $x \in [0,1] = (0,1) \cup \{0,1\}$.]

   Solution: $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in \{0,1\} \\ 0 & \text{if } x \in (0,1). \end{cases}$
   $= 0$ for each $x \in [0,1]$.

   (c) (6 points) Show that the convergence is uniform on $[0,1]$.

   Solution: For each $x \in [0,1]$, since $|f_n(x) - f(x)| = |f_n(x)| \leq f_n(\frac{n}{n+1}) = \dfrac{\sqrt{n}}{n+1}(1 - \frac{1}{n+1})^n = \dfrac{\sqrt{n}}{n+1}(1 - \frac{1}{n+1})^{n+1}(1 - \frac{1}{n+1})^{-1} \to 0$, the convergence is uniform on $[0,1]$.

10. Let f, g be **uniformly continuous** maps defined on $D \subset \mathbb{R}^p$ with ranges in $\mathbb{R}^q$.
   (a) Prove that $f + g$ is **uniformly continuous** on D.

   Solution: For each $\epsilon > 0$ since f, g are uniformly continuous on D, there exists a $\delta > 0$ such that if $x, y \in D$ and $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \epsilon$ and $\|g(x) - g(y)\| < \epsilon$
   $\Rightarrow \|(f+g)(x) - (f+g)(y)\| = \|f(x) - f(y) + g(x) - g(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| < 2\epsilon$.
   Hence, that $f + g$ is uniformly continuous on D.

   (b) If f and g are bounded on D (by M). Prove that the product fg is **uniformly continuous** on D.

   Solution: Assume that $\|f(x)\|, \|g(x)\| \leq M$ for each $x \in D$.
   Given $\epsilon > 0$ since f, g are uniformly continuous on D, there exists a $\delta > 0$ such that if $x, y \in D$ and $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \epsilon$ and $\|g(x) - g(y)\| < \epsilon$
   $\Rightarrow \|(fg)(x) - (fg)(y)\| = \|f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)\| \leq \|f(x) - f(y)\|\|g(x)\| + \|f(y)\|\|g(x) - g(y)\| < 2M\epsilon$.
   Hence, that fg is uniformly continuous on D.